The $(2,5)$ minimal model on degenerating genus two surfaces

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Abstract

In the $(2,5)$ minimal model, the partition function for genus $g = 2$ Riemann surfaces is expected to be given by a quintuplet of Siegel modular forms that extend the Rogers-Ramanujan functions on the torus. Their expansions around the $g = 2$ boundary components of the moduli space are obtained in terms of standard modular forms. In the case where a handle of the $g = 2$ surface is pinched, our method requires knowledge of the 2-point function of the fundamental lowest-weight vector in the non-vacuum representation of the Virasoro algebra, for which we derive a third order ODE.

1 Introduction

1.1 Motivation and outline

Two-dimensional conformal field theories (CFTs) are naturally defined on compact Riemann surfaces. Every such theory is characterised by its partition function, which defines a function on the moduli space of such surfaces. Its restriction to genus $g = 1$ is given by classical modular functions. For the $(2,5)$ minimal model, one obtains the sum of the squared norms of the well-known Rogers-Ramanujan functions. These 0-point functions satisfy a second order ODE in the modulus. For $g = 2$, a corresponding system of differential equations has been established in [6]. The method relies on the description of the Riemann surface Σ as a double covering of the Riemann sphere,

$$\Sigma : \quad y^2 = p(x)\,, \tag{1}$$

where p is a polynomial of degree 3 (for genus $g = 1$) resp. 5 (for $g = 2$).

A different method for computing N-point functions of CFTs on higher genus Riemann surfaces due to [8] is available, by sewing pairs of lower genus Riemann surfaces [9]. The case of interest to us in this paper is $N = 0$.

For $i = 1,2$, let (Σ_i, P_i) with $P_i \in \Sigma_i$ be a non-singular Riemann surface of genus g_i with puncture P_i. Let z_i be a local coordinate vanishing at P_i. We allow arbitrary complex coordinate choices. Now excise sufficiently small discs $\{|z_1| < r\}$ and $\{|z_2| < r\}$ from Σ_1 and Σ_2, respectively, and sew the two remaining surfaces by the condition

$$z_1 z_2 = r^2 \tag{2}$$

on tubular neigbourhoods of the circles $\{|z_i| = r\}$. This operation yields a non-singular Riemann surface of genus $g_1 + g_2$ with no punctures.

Instead of sewing two one-punctured surfaces, we may self-sew a single Riemann surface (the case $\Sigma_1 = \Sigma_2$) with two different punctures. This procedure results in a Riemann surface with one new handle attached to it.

Thus we consider the inverse procedure by which the genus $g = 2$ surface degenerates. Such singular surfaces are boundary points of the the moduli space with Deligne-Mumford compactification. In the limit where $r^2 \searrow 0$, a cycle on the surface is pinched. When the cycle is homologous to zero (case discussed in Section 2.1), the squeezing results in two separate tori with a single puncture on each. In the algebraic description by eq. (1), three ramification points run together. In the case where the cycle is non-homologous to zero (addressed in Section 2.2), the above mentioned limit describes the cut through a handle. In this case two ramification points run together, yielding a single torus with two punctures. To distinguish the two cases, following [7], we shall refer to the first and second case as the ε and the ρ formalism, respectively.

Using methods from vertex operator algebras, T. Gilroy and M. Tuite have derived the first terms of the corresponding expansion for the ε formalism [2]. In this paper, we give an expansion in terms of modular forms which in particular includes these earlier results.

One purpose of this paper is to built a bridge between the two approaches, and to make the subject better accessible to researchers interested in Siegel modular forms.

1.2 Quasi-primary and derivative fields

Let F be the space of holomorphic fields, (equivalently, the space of holomorphic states). A distinguished element in F is the Virasoro field

$$T(z) = \sum_{n\in\mathbb{Z}} z^{n-2} L_n \,.$$

The constant field 1 corresponds to the vacuum state v, the Virasoro field to $L_2 v$. The Laurent coefficients define the Virasoro algebra

$$[L_n, L_m] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0}\,, \tag{3}$$

where $c \in \mathbb{R}$ is the central charge. (Note the unusual index convention, which is chosen so that $L_1 = \frac{\partial}{\partial z}$). The kernel of L_1 is spanned by the vacuum vector v. L_0 defines a grading on F, called the conformal weight. Holomorphic fields in the image of L_1 will be referred to as derivative fields, whose space we denote by F_{der}. The Shapovalov metric defines a sesquilinear form on F. For the latter we have $L_{-1} = L_1^*$. The space of quasi-primary fields is the orthogonal complement of F_{der} w.r.t. that metric, thus the kernel of L_{-1}. A holomorphic field ψ is primary iff $L_n\psi = 0$ for $n < 0$. Suppose in some minimal model, W is an irreducible representation of the Virasoro algebra (3). Then there exists $w \in W$ with $L_0 w = hw$ and h is minimal in W. W is spanned by vectors of the form $L_{n_k} \dots L_{n_1} w$ with $n_k \in \mathbb{Z}$, $k \geq 0$. The vacuum representation is characterised by

$$L_{-1} v = 0\,, \quad L_0 v = 0\,, \quad L_1 v = 0\,.$$

The generating function of W is the character

$$\chi_W := \mathrm{tr}_{F_W} q^{L_0}\,.$$

Let $\tilde{F}_W$ be the space of quasi-primary fields in the representation W. If $w = v$, the generating function of $\tilde{F}_W$ is

$$\tilde{\chi}_W = (1-q)(\chi_W - 1)\,.$$

For other vectors one has

$$\tilde{\chi}_W = (1-q)\chi_W\,.$$

1.3 The $(2,5)$ minimal model

For every minimal model and for every irreducible representation of the Virasoro algebra, there are two fundamental linear relations between states in that representation. In the $(2,5)$ minimal model, the Virasoro algebra has two irreducible representations, the vacuum respresentation V (with vacuum vector v) and a non-vacuum representation which we denote by W. The lowest weight vector w in W has conformal weight $h = -1/5$. The fundamental identities in V are

$$\begin{aligned} L_1 v &= 0 \\ (L_2 L_2 - \frac{3}{5} L_4) v &= 0\,. \end{aligned}$$

Equivalently, the operator product expansion (OPE) of $T(z) \otimes T(0)$ has the form

$$T(z) \otimes T(0) \mapsto \frac{c/2}{z^4}.1 + \frac{1}{z^2}\{T(z) + T(0)\} + \frac{3}{10} T''(0) + O(z)\,, \tag{4}$$

where $c = -22/5$. The two fundamental identities in W are

$$(2L_2 - 5L_1 L_1) w = 0 \tag{5}$$

$$(L_3 - 5L_2 L_1) w = 0\,. \tag{6}$$

To w corresponds a non-holomorphic field Φ. For suitable pairs $(z, \bar{z})$ of a holomorphic and an antiholomorphic local coordinate the field's local representative admits a splitting $\Phi(z,\bar{z}) = \varphi_{\mathrm{hol}}(z) \otimes \varphi_{\overline{\mathrm{hol}}}(\bar{z})$ into holomorphic and antiholomorphic part. The individual holomorphic part $\varphi = \varphi_{\mathrm{hol}}$ is a local primary field of conformal weight $h = -1/5$. Thus eqs (5) and (6) are equivalent to the OPE

$$T(z) \otimes \varphi(0) \mapsto \frac{h}{z^2}\varphi(0) + \frac{1}{z}\varphi'(0) + \frac{5}{2}\varphi''(0) + \frac{25}{12}\, z\varphi^{(3)}(0) + O(z^2)\,, \tag{7}$$

where $h = -1/5$. The space of all fields factorises as

$$F = F_V \otimes \overline{F_V} \oplus F_W \otimes \overline{F_W}\,,$$

where F_V and F_W denote the space of holomorphic fields that correspond to states in V and W, respectively, and the bar marks the corresponding spaces of antiholomorphic fields.

For the $(2,5)$ minimal model, the generating function for the number of holomorphic fields of a given weight in F_V and in F_W is the character

$$\begin{aligned}
\chi_V &= \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} \\
&= 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + 2q^7 + 3q^8 + 3q^9 + 4q^{10} + 4q^{11} + 6q^{12} + \ldots\,, \\
\chi_W &= q^{-\frac{1}{5}} \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} \\
&= q^{-\frac{1}{5}} \left(1 + q + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 5q^9 + 6q^{10} + 7q^{11} + 9q^{12} + \ldots\right)\,,
\end{aligned}$$

respectively. (Here $(\ ;\ )_n$ is the q-Pochhammer symbol.)

Propos. 1. *To every conformal weight $h \leq 10$, there exists at most one quasi-primary field in F_V, up to normalisation. For $h \leq 8$, their respective weight and squared Shapovalov norm are given by the following table:*

weight	*quasi-primary field*	*squared norm*
2	$L_2 v$	$c/2$
4	-	-
6	$(7L_4L_2 - 2L_6)v$	$217c$
8	$(6L_6L_2 + \frac{21}{5}L_4L_4 - 7L_8)v$	$-\frac{8952}{5}c$

Here $c = -22/5$.

Proof. The number of quasi-primary fields of conformal weight h in F_V is given by the coefficient of q^h in the series

$$\tilde{\chi}_V - 1 = (1-q)(\chi_V - 1) = q^2 + q^6 + q^8 + q^{10} + 2q^{12} + q^{15} + \ldots\,.$$

The fields and their respective weight are obtained by direct computation. □

Propos. 2. *To every conformal weight $h \leq 11$, there exists at most one quasi-primary field in F_W, up to normalisation. For $h < 6$, their respective weight and squared Shapovalov norm are given by the following table:*

weight		*quasi-primary field*	*squared norm*
$-\frac{1}{5}$	$+2$	$-$	$-$
	$+4$	$(52L_4 - 25L_1L_3)w$	$\frac{5928}{5} \parallel w \parallel^2$
	$+6$	$(4L_1L_5 + 3L_3L_3 - \frac{684}{35}L_6)w$	$\frac{6539268}{6125} \parallel w \parallel^2$

Proof. The number of quasi-primary fields of conformal weight h in F_W is given by the coefficient of q^h in the series

$$\tilde{\chi}_W = (1-q)\chi_W = q^{-\frac{1}{5}} \left(1 + q^4 + q^6 + q^8 + q^9 + q^{10} + q^{11} + 2q^{12} + \ldots\right)$$

The fields and their respective weight are obtained by direct computation. □

Now we specialise to $g = 1$. The 0-point functions differ from the corresponding characters by a factor of $q^{-\frac{c}{24}}$, where q is identified with the nome $e^{2\pi i\tau}$. For the $(2,5)$

minimal model on the torus, these are the so-called Rogers-Ramanujan functions

$$\langle 1\rangle_1^{g=1}(q) = H(q) := q^{\frac{11}{60}} \sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}\,,$$

$$\langle 1\rangle_2^{g=1}(q) = G(q) := q^{-\frac{1}{60}} \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}\,.$$

The modular invariant partition function is given by

$$Z^{g=1}(q) = |H(q)|^2 + |G(q)|^2\,.$$

The Virasoro field generates changes of τ, so that ([1], or [5] for a direct proof)

$$\mathcal{D}_0\langle 1\rangle = \frac{1}{(2\pi i)^2}\langle T\rangle\,. \tag{8}$$

As an aside, the OPE (4) yields in addition

$$\mathcal{D}_2\langle T\rangle = \frac{11}{3600}(2\pi i)^2 E_4(q)\langle 1\rangle\,.$$

Here for $\ell \in \mathbb{R}$,

$$\mathcal{D}_\ell = q\frac{\partial}{\partial q} - \frac{\ell}{12}E_2(q)$$

is the Serre-derivative operator, (defined on modular forms of weight ℓ).

Let $\wp(z|\tau)$ and $\zeta(z|\tau)$ (or $\wp(z)$ and $\zeta(z)$ when $\tau \in \mathbb{H}^+$, the upper complex half plane, is fixed) be the Weierstrass $\wp$-function and the Weierstrass ζ-function, respectively. For brevity, we write $\wp_{ij}$ and ζ_{ij} in place of $\wp(z_{ij})$ and $\zeta(z_{ij})$ respectively, where for $z_i, z_j \in \mathbb{C}$, $z_{ij} := z_i - z_j$.

Now we calculate the 1-point function of the field $\varphi \in F_W$ corresponding to the lowest weight vector w in W.

Propos. 3. *We have*

$$\mathcal{D}_{-1/5}\langle\varphi\rangle = 0\,.$$

Proof. By the OPE (7),

$$\frac{\langle T(z)\varphi(0)\rangle}{\langle\varphi(0)\rangle} = -\frac{1}{5}\wp(z)\,. \tag{9}$$

Indeed, the regular part must be constant, and is zero by the fact that $\langle\varphi''(0)\rangle = \partial^2\langle\varphi(0)\rangle = 0$. Thus

$$q\frac{d}{dq}\langle\varphi(0)\rangle = \oint\langle T(z)\varphi(0)\rangle\frac{dz}{(2\pi i)^2} = -\frac{1}{5}\langle\varphi(0)\rangle\int_0^1 \wp(z)\frac{dz}{(2\pi i)^2} = -\frac{1}{60}E_2(q)\langle\varphi(0)\rangle\,.$$

(Here the contour integral is taken along the real period, and $\oint dz = 1$). □

Thus we have

$$\begin{aligned}\langle\varphi\rangle = \eta(q)^{-2/5} &= q^{-\frac{1}{60}}\prod_{n\geq 1}(1-q^n)^{-2/5}\\ &= q^{-\frac{1}{60}}\left(1 + \frac{2}{5}q + \frac{17}{25}q^2 + \frac{98}{125}q^3 + \frac{714}{625}q^4 + \frac{18,768}{15,625}q^5 + \ldots\right).\end{aligned}$$

Here $\eta(q)$ is Dedekind η-function.

Corollary 4. *Set $z_0 = 0$. We have*

$$\frac{\langle T(z_1)T(z_2)\varphi(0)\rangle}{\langle\varphi(0)\rangle} = \frac{c}{2}\wp_{12}^2 - \frac{1}{5}\wp_{12}\left(\wp_{10} + \wp_{20}\right) + \frac{6}{25}\wp_{10}\wp_{20} + \frac{\pi^4}{45}E_4\,. \tag{10}$$

Proof. On the one hand, from the OPE (7) for $T(z)\otimes\varphi(0)$ and eq. (9),

$$\eta^{2/5}\langle T(z)T(w)\varphi(0)\rangle = \frac{h^2}{z^2}\wp(w) - \frac{h}{z}\wp'(w) + \text{terms that are regular for } z \to 0\,,$$

where the occurring even and odd negative power of z can be replaced with $\wp(z)$ and $z\wp(z)$, respectively. The latter expression is not elliptic, however we may use

$$-z\wp'(w) = \wp(z-w) - \wp(w) + O(z^2)\,.$$

Thus we have

$$\begin{aligned}&\eta^{2/5}\langle T(z)T(w)\varphi(0)\rangle\\&\quad= h\wp(z)\wp(z-w) + (h^2-h)\wp(z)\wp(w) + \text{terms that are regular for } z \to 0\,.\end{aligned} \tag{11}$$

On the other hand, by the OPE (4) for $T(z)\otimes T(w)$, using eq. (9),

$$\eta^{-2/5}\langle T(z)T(w)\varphi(0)\rangle = \frac{c/2}{(z-w)^4} + \frac{h}{(z-w)^2}\{\wp(z)+\wp(w)\} + \frac{3}{10}h\wp''(w) + O(z-w)\,.$$

Thus

$$\begin{aligned}&\eta^{2/5}\langle T(z)T(w)\varphi(0)\rangle\\&\quad= \frac{c}{12}\wp''(z-w) + h\wp(z-w)\{\wp(z)+\wp(w)\} + (h^2-h)\wp(z)\wp(w) + K\,,\end{aligned} \tag{12}$$

where K is constant in z and w. Comparison of eqs (11) and (12) yields

$$h\left(h+\frac{1}{5}\right) = 0\,,\quad K = -(c+10h(h-1))\frac{\pi^4}{90}E_4 = \frac{\pi^4}{45}E_4\,.$$

□

2 The genus 2 partition function

2.1 Results in the ε formalism

Let $\{\psi_i\}_{i\geq 0}$ be an orthonormal basis of F_V with the Shapovalov metric, where $\psi_0 = 1$ and $L_0\psi_i = h_i\psi_i$ for $i \geq 0$. For any $\psi \in F_V$, denote by $\psi(z)$ and $\hat{\psi}(\hat{z})$ the local representative of ψ w.r.t. a chart of an affine structure [3] on the torus with modulus τ and $\hat{\tau}$, respectively. In the respective coordinate z and $\hat{z}$, all 1-point functions on either torus are constant in position. On a small annulus centered at $z = 0$ resp. $\hat{z} = 0$, we glue the two tori using

$$z\,\hat{z} = \varepsilon$$

for $\varepsilon > 0$ small. This procedure yields a $g = 2$ surface with a projective structure. Let $\tilde{z} = \hat{z}/\varepsilon$ and write $\tilde{\psi}(\tilde{z})$ accordingly. For $a, b \in \{1, 2\}$, the choice of the Roger-Ramanujan function $\langle 1\rangle_a^{g=1}$ and $\langle 1\rangle_b^{g=1}$ on the torus of modulus τ and $\hat{\tau}$, respectively, gives rise to the 0-point function for the index pair (a, b)

$$\langle 1\rangle_{a,b}^{g=2}(q, \hat{q}, \varepsilon) = \sum_{i\geq 0} \langle\psi_i(z)\rangle_a^{g=1}(q)\, \langle\tilde{\psi}_i(\tilde{z})\rangle_b^{g=1}(\hat{q}) \tag{13}$$

on the resulting genus $g = 2$ surface. A fifth solution $\langle 1\rangle_{\varphi}^{g=2}(q, \hat{q}, \varepsilon)$ is obtained by choosing $\langle\varphi\rangle$ on either torus.

For $i > 0$, only the ψ_i that are quasi-primary contribute to the sum. Under the coordinate change $\tilde{z} \mapsto \hat{z}$, the 1-point functions transform according to

$$\langle\tilde{\psi}_i(\tilde{z})\rangle = \varepsilon^{h_i}\langle\hat{\psi}_i(\hat{z})\rangle\,,$$

so eq. (13) becomes an infinite series in powers of ε. We will use the notation

$$\langle\psi_i\rangle = \langle\psi_i\rangle^{g=1}(q)\,, \quad \widehat{\langle\psi_i\rangle} = \langle\hat{\psi}_i\rangle^{g=1}(\hat{q})\,.$$

We also write $E_{2k} = E_{2k}(q)$ and $\widehat{E}_{2k} = E_{2k}(\hat{q})$ and likewise for other modular forms.

Theorem 1. *For $a = 1, 2$ we have*

$$\begin{aligned}\langle 1\rangle_{a,a}^{g=2}(q, \hat{q}, \varepsilon) = F_0\widehat{F}_0 &+ \frac{\varepsilon^2}{7920}(2\pi i)^4 F_2\widehat{F}_2 + \frac{\varepsilon^6}{445,471,488,000}(2\pi i)^{12} F_6\widehat{F}_6 \\ &- \frac{\varepsilon^8}{125,067,317,760,000}(2\pi i)^{16} F_8\widehat{F}_8 + O(\varepsilon^{10})\,,\end{aligned}$$

Here $F_{2k} = F_{2k}(q)$ and $\widehat{F}_{2k} = F_{2k}(\hat{q})$ are given by

$$\begin{aligned} F_0 &= \langle 1\rangle_a\,, \quad F_2 := 60q\frac{\partial}{\partial q}F_0 = \frac{60}{(2\pi i)^2}\langle T\rangle_a\,, \quad a = 1, 2; \\ F_6 &:= 110E_6F_0 + 21E_4F_2\,, \\ 6F_8 &:= 1309E_8F + 235E_6F_2\,. \end{aligned}$$

For $F_0 = H$, the expansions

$$F_2(q) = q^{11/60}\left(11 + 131q^2 + 191q^3 + 251q^4 + 311q^5 + 742q^6 + 862q^7 + 1473q^9 + O(q^{10})\right)$$
$$F_6(q) = q^{11/60}\left(341 - 1,327,699q^2 - 11,366,119q^3 - 49527739q^4 - 153310159q^5 - 418324358q^6 + O(q^7)\right)$$
$$F_8(q) = q^{11/60}\left(649 - 112,420q + 6,348,609q^2 + 173,671,679q^3 + 1,424,241,669q^4 + O(q^5)\right)$$

(and similar expansions for $F_0 = G$) have been found previously by [2] though the coefficients have not been identified with standard modular forms.

Proof. According to Proposition 1, we have for $a, b \in \{1, 2\}$,

$$\begin{aligned}\langle 1\rangle_{a,b}^{g=2}(q, \hat{q}, \varepsilon) = \langle 1\rangle_a\widehat{\langle 1\rangle}_b &- \frac{2}{c}\varepsilon^2\langle T\rangle_a\widehat{\langle T\rangle}_b - \frac{7}{31c}\varepsilon^6\langle L_4L_21\rangle_a\widehat{\langle L_4L_21\rangle}_b \\ &- \frac{5\varepsilon^8}{8952c}\langle(6L_6L_2 + \frac{21}{5}L_4L_4)v\rangle_a\widehat{\langle(6L_6L_2 + \tfrac{21}{5}L_4L_4)v\rangle}_b + O(\varepsilon^{10})\,.\end{aligned}$$

We have

$$\langle T(z)T(0)\rangle = \sum_{n\in\mathbb{Z}} z^{n-2}\langle L_n L_2 v\rangle \,.$$

Comparison with

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2}\wp_{12}^2\langle 1\rangle + 2\wp_{12}\langle T\rangle - \frac{\pi^2}{15}E_4 c\langle 1\rangle$$

[4, and references therein] yields:

$$\begin{aligned}
\langle L_4 L_2 v\rangle &= \frac{2c}{189}\pi^6 E_6\langle 1\rangle + \frac{2}{15}\pi^4 E_4\langle T\rangle \\
\langle L_6 L_2 v\rangle &= \frac{c}{270}\pi^8 E_8\langle 1\rangle + \frac{4}{189}\pi^6 E_6\langle T\rangle \\
\langle L_4 L_4 v\rangle &= \frac{7c}{315}\pi^8 E_8\langle 1\rangle + \frac{48}{189}\pi^6 E_6\langle T\rangle
\end{aligned}$$

We conclude that for the other quasi-primary fields listed in Proposition 1, we have

$$\begin{aligned}
\langle (7L_4L_2 - 2L_6)v\rangle_1 &= (2\pi i)^6 \frac{F_6}{21600} \\
\langle (6L_6L_2 + \frac{21}{5}L_4L_4 - 7L_8)v\rangle_1 &= -(2\pi i)^8 \frac{1309E_8F_0 + 235E_6F_2}{756000} \,.
\end{aligned}$$

□

In order to compute the higher order terms (i.e., the one-point function of quasi-primary fields of conformal weight $h \geq 12$), N-point functions for $N \geq 3$ are required.

Theorem 2. *[4] Let $S(z_1, \ldots, z_N)$, $N \in \mathbb{N}$, be the set of oriented graphs with vertices $z_1, \ldots, z_N$ (which may or may not be connected), subject to the condition that every vertex has at most one ingoing and at most one outgoing line, and none is a tadpole (with the incoming line being identical to the outgoing line). We have*

$$\langle T(z_1)\ldots T(z_N)\rangle^{g=1} = \sum_{\Gamma\in S(z_1,\ldots,z_N)} F(\Gamma)\,,$$

where for $\Gamma \in S(z_1, \ldots, z_N)$,

$$F(\Gamma) := \left(\frac{c}{2}\right)^{\sharp loops} \prod_{(z_i,z_j)\in\Gamma} \wp_{ij} \left\langle \bigotimes_{k\in E_N^c} T(z_k)\right\rangle_r \,.$$

Here $(z_i, z_j) \in \Gamma$ is an oriented edge,

$$E_N := \{1 \leq j \leq N \,|\, \exists\, i \text{ such that } (z_i, z_j) \in \Gamma\}\,,$$

and E_N^c denotes its complement in $\{1, \ldots N\}$. Moreover, for all $n \in \mathbb{N}$, $\langle T(z_{k_1}) \otimes \ldots \otimes T(z_{k_n})\rangle_r$ with $k_i \in E_N^c$ for $i = 1, \ldots, n$, is a modular form of weight $2n$.

Example 5. *For $\langle 1\rangle = \langle 1\rangle_a^{g=1}$ with $a \in \{1,2\}$, and for $\langle T\rangle$ given by eq. (8), we have*

$$\begin{aligned}
\langle 1\rangle_r &= \langle 1\rangle \\
\langle T(z)\rangle_r &= \langle T\rangle \\
\langle T(z_1)T(z_2)\rangle_r &= -\frac{\pi^4}{15}E_4 c\langle 1\rangle \\
\langle T(z_1)T(z_2)T(z_3)\rangle_r &= -\frac{4\pi^6}{45}E_6 c\langle 1\rangle + \frac{14\pi^4}{25}E_4\langle T\rangle \\
\langle T(z_1)T(z_2)T(z_3)T(z_4)\rangle_r &= -\frac{1,468\pi^8}{10,125}E_4^2 c\langle 1\rangle + \frac{1,792\pi^6}{1,575}E_6\langle T\rangle\,.
\end{aligned}$$

We discuss the fifth solution, which is characterised by properties of φ.

Theorem 3. *We have*

$$\begin{aligned}
\langle 1\rangle_\varphi^{g=2}(q,\hat q,\varepsilon) = \varepsilon^{-1/5}\,(\eta\,\widehat{\eta})^{-2/5}\Big\{1 &+ \frac{13}{8,208,000}(2\pi i)^8\varepsilon^4 E_4\widehat{E}_4 \\
&+ \varepsilon^6(2\pi i)^{12}\frac{989}{33,591,075,840}E_6\widehat{E}_6 + O(\varepsilon^8)\Big\}\,.
\end{aligned}$$

Proof. By Proposition 2,

$$\begin{aligned}
\langle 1\rangle_\varphi^{g=2}(q,\hat q,\varepsilon) = \varepsilon^{-1/5}\Big\{&\langle\varphi\rangle\widehat{\langle\varphi\rangle} + \frac{5\cdot(52)^2}{5,928}\varepsilon^4\langle L_4 w\rangle\widehat{\langle L_4 w\rangle} \\
&+ \varepsilon^6\frac{6,125}{6,539,268}\langle(3L_3L_3 - \frac{684}{35}L_6)w\rangle\widehat{\langle(3L_3L_3 - \tfrac{684}{35}L_6)w\rangle} + O(\varepsilon^8)\Big\}\,.
\end{aligned}$$

We list the partial results: By eq. (9) we have

$$\begin{aligned}
\langle L_4 w\rangle &= -\frac{(2\pi i)^4}{1200}E_4\eta^{-2/5} \\
\langle L_6 w\rangle &= \frac{(2\pi i)^6}{30,240}E_6\eta^{-2/5}\,.
\end{aligned}$$

Sorting out the terms $\propto z_1 z_2$ in eq. (10) yields:

$$\langle L_3L_3 w\rangle = \frac{5}{3,024}(2\pi i)^6 E_6\eta^{-2/5}\,.$$

We conclude that for the quasi-primary fields listed in Proposition 2, we have

$$\begin{aligned}
\langle(52L_4 - 25L_1L_3)w\rangle &= -(2\pi i)^4\frac{13}{300}E_4\eta^{-2/5} \\
\langle(4L_1L_5 + 3L_3L_3 - \frac{684}{35}L_6)w\rangle &= -(2\pi i)^6\frac{989}{176,400}E_6\eta^{-2/5}\,.
\end{aligned}$$

□

The $g=2$ partition function is

$$Z^{g=2} = \sum_{a,b=1,2}|\langle 1\rangle_{a,b}^{g=2}|^2 + \lambda|\langle 1\rangle_\varphi^{g=2}|^2\,,$$

where $\lambda \in \mathbb{R}$ is such that $Z^{g=2}$ is modular.

2.2 Results in the ρ formalism

We consider a torus with modulus τ and two punctures separated from one another by a pair of disjoint neighbourhoods with local coordinates z_1 and z_2 respectively, which vanish at the respective puncture. The torus is self-sewed by imposing the condition

$$z_1\, z_2 = \rho$$

for some $\rho > 0$. For any of the two local coordinates, we define $\tilde{z} = z/\rho$. This gives rise to local representatives of a field ψ denoted by $\tilde{\psi}(\tilde{z})$. Let $\{\psi_i\}_{i\geq 0}$ be an orthonormal basis of F_V with the Shapovalov metric, where $\psi_0 = 1$ and $L_0\psi_i = h_i\psi_i$ for $i \geq 0$. For $k \geq 0$, we have

$$\frac{2}{c} \parallel \partial^k T \parallel^2 = \frac{k!(k+3)!}{3!} = 1, \frac{1}{4}, \frac{1}{40}, \frac{1}{720}, \frac{1}{20,160}, \ldots .$$

The choice of a Rogers Ramanujan function $\langle 1\rangle_a^{g=1}$ on the torus with $a \in \{1,2\}$ gives rise to a 0-point function

$$\langle 1\rangle_a^{g=2} = \sum_{i\geq 0} \langle \psi_i(z_1)\tilde{\psi}_i(\tilde{z}_2)\rangle_a^{g=1} . \tag{14}$$

for genus $g = 2$. Since

$$\tilde{\partial}^k \tilde{T}(\tilde{z}) = \rho^{2+k}\partial^k T(z)\,, \quad k \geq 0\,,$$

where $\partial = \partial/\partial_z$ and $\tilde{\partial} = \partial/\partial_{\tilde{z}}$, eq. (14) becomes an expansion in powers of ρ,

$$\langle 1\rangle_a^{g=2} = \sum_{i\geq 0} \rho^{h_i} \langle \psi_i(z_1)\psi_i(z_2)\rangle_a^{g=1} . \tag{15}$$

Note that upon setting three ramification points equal to $0, 1, \infty$, respectively, for every choice of a, either side of the equation depends on three parameters. On the l.h.s. we have the remaining three ramification points for genus $g = 2$. On the r.h.s., we are free to choose the difference $z_1 - z_2$, the perturbation parameter ρ and the modulus τ (or the remaining ramification point) of the torus.

For the vacuum sector, the first non-trivial term in the series of eq. (15) occurs for weight $h = 6$.

Propos. 6. *We have*

$$\langle (7L_4L_2 - 2L_6)(z_1)(7L_4L_2 - 2L_6)(z_2)\rangle = 7\,\{12 g(z_{12})\langle T\rangle + f(z_{12})c\langle 1\rangle\}\,,$$

where

$$\begin{aligned} f(z_{12}) = &-151,831\wp_{12}^6 + \frac{303,662}{5}\pi^4 E_4\wp_{12}^4 + \frac{1,813,300}{189}\pi^6 E_6\wp_{12}^3 - \frac{71,057}{15}\pi^8 E_4^2\wp_{12}^2 \\ &- \frac{1,046,828}{945}\pi^{10}E_4E_6\wp_{12} - \frac{5,768}{135}\pi^{12}E_6^2 + \frac{1,706}{125}\pi^{12}E_4^3\,, \end{aligned}$$

and

$$g(z_{12}) = 5,765\wp_{12}^5 - \frac{88,643}{45}\pi^4 E_4\wp_{12}^3 - \frac{294,326}{945}\pi^6 E_6\wp_{12}^2 + \frac{4,192}{45}\pi^8 E_4^2\wp_{12} + \frac{77,542}{4725}\pi^{10}E_4E_6.$$

Proof. Using the contour integral method and sorting out the coefficient of z^4w^4 in $\langle T(z+z_1)T(w+z_2)\rangle$ yields

$$\begin{aligned}\langle L_6(z_1)L_6(z_2)\rangle = \Big\{&\frac{400}{243}\,\pi^{12}E_6^2 - \frac{14}{27}\,\pi^{12}E_4^3\\ &+ \frac{380}{9}\,\pi^{10}E_4E_6\wp_{12} + \frac{539}{3}\,\pi^8E_4^2\wp_{12}^2 - \frac{1,100}{3}\,\pi^6E_6\wp_{12}^3 - 2,310\,\pi^4E_4\wp_{12}^4 + 5,775\wp_{12}^6\Big\}c\langle 1\rangle\\ &+\Big\{\frac{88}{27}\,\pi^{10}E_4E_6 + \frac{56}{3}\,\pi^8E_4^2\wp_{12} - \frac{200}{3}\,\pi^6E_6\wp_{12}^3 - 420\,\pi^4E_4\wp_{12}^3 + 1,260\wp_{12}^5\Big\}\langle T\rangle\,.\end{aligned}$$

Sorting out the coefficient of $(z-z_1)^2$ in $\langle T(z)T(z_1)T^{(4)}(z_2)\rangle$ yields

$$\begin{aligned}\langle L_4L_2(z_1)L_6(z_2)\rangle = \Big\{&\frac{896}{81}\,\pi^{12}E_6^2 - \frac{32}{9}\,\pi^{12}E_4^3\\ &+ 288\,\pi^{10}E_4E_6\wp_{12} + 1,232\,\pi^8E_4^2\wp_{12}^2 - \frac{7,520}{3}\,\pi^6E_6\wp_{12}^3 - 15,840\,\pi^4E_4\wp_{12}^4 + 39,600\wp_{12}^6\Big\}c\langle 1\rangle\\ &+\Big\{-\frac{448}{9}\,\pi^{10}E_4E_6 - 288\,\pi^8E_4^2\wp_{12} + 960\,\pi^6E_6\wp_{12}^2 + 6,144\,\pi^4E_4\wp_{12}^3 - 18,144\wp_{12}^5\Big\}\langle T\rangle\,.\end{aligned}$$

Sorting out the coefficient of $(z-z_1)^2(w-z_2)^2$ in $\langle T(z)T(z_1)T(w)T(z_2)\rangle$ yields

$$\begin{aligned}\langle L_4L_2(z_1)L_4L_2(z_2)\rangle = \Big\{&\frac{4,936}{59,535}\pi^{12}\,E_6^2 - \frac{134}{3,375}\,\pi^{12}E_4^3\\ &+ \frac{388}{135}\,\pi^{10}E_4E_6\wp_{12} + \frac{19}{15}\,\pi^8E_4^2\wp_{12}^2 - \frac{860}{27}\,\pi^6E_6\wp_{12}^3 - \frac{934}{5}\,\pi^4E_4\wp_{12}^4 + 467\wp_{12}^6\Big\}c\langle 1\rangle\\ &+\Big\{\frac{2,728}{4,725}\,\pi^{10}E_4E_6 + \frac{32}{5}\,\pi^8E_4^2\wp_{12} + \frac{904}{45}\,\pi^6E_6\wp_{12}^2 + \frac{2,524}{15}\,\pi^4E_4\wp_{12}^3 - 588\wp_{12}^5\Big\}\langle T\rangle\,.\end{aligned}$$

From this follows the claimed equation. □

We list the first few terms in eq. (15).

k	coefficient of ρ^k/c
0	$c\langle 1\rangle$
2	$4P_1\langle T\rangle + (P_2 - \frac{1}{90}E_4\pi^4)c\langle 1\rangle$
3	$-6P_2\langle T\rangle - 5P_3c\langle 1\rangle$
4	$12P_3\langle T\rangle + 21P_4c\langle 1\rangle$
5	$28P_4\langle T\rangle - 84P_5c\langle 1\rangle$
6	$\frac{1}{217}\langle(7L_4L_2 - 2L_6)(z_1)(7L_4L_2 - 2L_6)(z_2)\rangle + 72P_5\langle T\rangle + 330P_6c\langle 1\rangle$

where $\langle 1\rangle = \langle 1\rangle_a^{g=1}$, $a\in\{1,2\}$, and $\langle T\rangle$ is is given by eq. (8). The P_i are polynomials in $\wp = \wp_{12}$ defined by

$$\begin{aligned}P_1 =&\; \wp\\ P_2 =&\; \wp^2 - \tfrac{1}{9}E_4\pi^4\\ P_3 =&\; \wp^3 - \tfrac{1}{5}E_4\pi^4\wp - \tfrac{4}{135}E_6\pi^6\\ P_4 =&\; \wp^4 - \tfrac{4}{15}E_4\pi^4\wp^2 - \tfrac{8}{189}E_6\pi^6\wp + \tfrac{1}{135}E_4^2\pi^8\\ P_5 =&\; \wp^5 - \tfrac{1}{3}E_4\pi^4\wp^3 - \tfrac{10}{189}E_6\pi^6\wp^2 + \tfrac{2}{135}E_4^2\pi^8\wp + \tfrac{22}{8,505}E_4E_6\pi^{10}\\ P_6 =&\; \wp^6 - \tfrac{2}{5}E_4\pi^4\wp^4 - \tfrac{4}{63}E_6\pi^6\wp^3 + \tfrac{11}{495}E_4^2\pi^8\wp^2 + \tfrac{76}{10,395}E_4E_6\pi^{10}\wp\\ &-\tfrac{2}{22,275}E_4^3\pi^{12} + \tfrac{16}{56,133}E_6^2\pi^{12}\,.\end{aligned}$$

For $N \geq 2$, N-point functions involving $\varphi(z)$ can be properly defined for $\Phi(z, \bar{z})$ only. On the torus, they may fail to be elliptic in z. In order to deal with this problem, we assume that z takes on a fixed value, or varies little about a fixed value. We show below that $\langle\varphi(z)\varphi(0)\rangle$ satisfies a third order ODE in z, so $\Phi(z, \bar{z})$ defines a 3-dimensional representation of the lattice translation group. In order to continue eq. (15) to $a = 3, 4, 5$, we must assume that $\langle\varphi(z_1)\varphi(z_2)\rangle_a$ is translationally invariant. In particular, $\langle\varphi(z)\varphi(0)\rangle_a$ is an even function of z.

Propos. 7. *Let $z_0 = 0$ and let z_2 be fixed. We have*

$$\begin{aligned}\langle T(z_1)\varphi(z_2)\varphi(0)\rangle^{g=1} &= h\{\wp_{12} + \wp_{10} - \wp_{20}\}\langle\varphi(z_2)\varphi(0)\rangle \\ &\quad + \{\zeta_{01} + \zeta_{12} + \zeta_{20}\}\langle\varphi'(z_2)\varphi(0)\rangle + \frac{5}{2}\langle\varphi''(z_2)\varphi(0)\rangle\,.\end{aligned}$$

Proof. By the OPE of $T(u) \otimes \varphi(z)$ and $T(u) \otimes \varphi(0)$, respectively,

$$\begin{aligned}\langle T(u)\varphi(z)\varphi(0)\rangle &= h\wp(u-z)\langle\varphi(z)\varphi(0)\rangle + \zeta(u-z)\langle\varphi'(z)\varphi(0)\rangle + \text{regular for } u \to z \\ &= h\wp(u)\langle\varphi(z)\varphi(0)\rangle + \zeta(u)\langle\varphi(z)\varphi'(0)\rangle + \text{regular for } u \to 0\end{aligned}$$

The Weierstrass zeta function fails to be periodic w.r.t. the torus periods ω_1, ω_2 but satisfies

$$\zeta(z + m\omega_1 + n\omega_2) - \zeta(z) = 2m\,\zeta(\frac{\omega_1}{2}) + 2n\,\zeta(\frac{\omega_2}{2})\,, \quad m, n \in \mathbb{Z}\,.$$

Thus the difference $\zeta(u-z) - \zeta(u)$ defines an elliptic function of u, while the sum does not. It follows that we necessarily have[1]

$$\langle\varphi(z)\varphi'(0)\rangle = -\langle\varphi'(z)\varphi(0)\rangle\,.$$

So

$$\begin{aligned}\langle T(u)\varphi(z)\varphi(0)\rangle &= h\{\wp(z-u) + \wp(u)\}\langle\varphi(z)\varphi(0)\rangle + \{\zeta(u-z) - \zeta(u)\}\langle\varphi'(z)\varphi(0)\rangle \\ &\quad + \text{terms that are regular in } u.\end{aligned} \tag{16}$$

In order for $\langle T(u)\varphi(z)\varphi(0)\rangle$ to be elliptic in u, the terms regular in u must actually be constant. Comparison of the u^0 terms in line (16) with the OPE (7) for $T(u) \otimes \varphi(0)$ shows that the terms constant in u are equal to

$$\frac{5}{2}\langle\varphi(z)\varphi''(0)\rangle - h\wp(z)\langle\varphi(z)\varphi(0)\rangle + \zeta(z)\langle\varphi'(z)\varphi(0)\rangle$$

(ζ is an odd function). Comparison with the terms constant in u which are obtained from the OPE for $T(u) \otimes \varphi(z)$ shows that

$$\langle\varphi''(z)\varphi(0)\rangle = \langle\varphi(z)\varphi''(0)\rangle\,.$$

□

Corollary 8. *Let $z_0 = 0$ and $z_1 = z$. The two-point function of φ satisfies the ODE*

$$\frac{25}{12}\frac{d^3}{dz^3}\langle\varphi(z)\varphi(0)\rangle^{g=1} = h\wp_{10}'\langle\varphi(z)\varphi(0)\rangle + \wp_{10}\langle\varphi'(z)\varphi(0)\rangle\,, \tag{17}$$

where $h = -1/5$.

[1] Alternatively, this follows from the assumption that $\langle\varphi(z)\varphi(0)\rangle$ is translationally invariant.

Proof. This follows from comparing the terms in line (16) which are linear in u with the OPE (7) for $T(u) \otimes \varphi(0)$, using that $\zeta'(z) = -\wp(z)$ for $z \in \mathbb{C}$, and the fact that $\langle \varphi^{(3)}(z)\varphi(0)\rangle = -\langle \varphi^{(3)}(0)\varphi(z)\rangle$. □

Using that for $k \geq 0$,

$$\frac{\parallel \partial^k \varphi \parallel^2}{\parallel \varphi \parallel^2} = k! \prod_{n=0}^{k-1} (k - n - \frac{7}{5}) \in \left\{ 1, -\frac{5}{2}, -\frac{25}{12}, -\frac{125}{288} \ldots \right\} ,$$

solving eq. (17) will allow to compute the coefficients of $\rho^{k-1/5} / \parallel \varphi \parallel^2$ in eq. (15). For example, $\langle L_4\varphi(z_1)L_1L_3\varphi(z_2)\rangle$ sorts out the coefficient proportional to $(z - z_1)^2(u - z_2)^{-1}(v - z_2)$ in $\langle T(z)T(u)T(v)\varphi(z_1)\varphi(z_2)\rangle$.

2.3 Outlook

Using the Frobenius Ansatz $\langle \varphi(z)\varphi(0)\rangle \sim z^\alpha$, the differential equation (17) imposes the condition

$$\frac{25}{12}\alpha(\alpha - 1)(\alpha - 2) = \frac{2}{5} + \alpha .$$

on α, which produces the values $1/5, 2/5$ and $12/5$. The obvious solutions to the ODE are, to leading order,

$$z^{1/5}\langle \varphi \rangle , \quad z^{2/5}\langle 1 \rangle ,$$

but the third exponent remains to be understood.

Remark 9. *Solving the ODE (17) is equivalent to solving the ODE*

$$y^{4/5}\left(p(x)\frac{d^3}{dx^3} + f(x)\frac{d^2}{dx^2} + g(x)\frac{d}{dx} + h(x) \right)\Psi(x) = 0 ,$$

where

$$p(x) = 4\left(x^3 - \frac{\pi^4}{3}E_4 x - \frac{2}{27}\pi^6 E_6 \right) ,$$

and

$$\begin{aligned} f &= \frac{6}{5}p' \\ g &= \frac{3}{100}\frac{[p']^2}{p} + \frac{9}{50}p'' \\ h &= -\frac{33}{500}\frac{[p']^3}{p^2} + \frac{33}{250}\frac{p'p''}{p} - \frac{288}{125} . \end{aligned}$$

In particular, the ODE has simple poles at the four ramification points.

Proof of the Remark. We change to the algebraic coordinates $x = \wp(z)$ and $y = \wp'(z)$ with $y^2 = p(x)$. Let $\check{\varphi}(x)$ be the local representative of φ and let $\Psi(x) = \langle \check{\varphi}(x)\varphi(0)\rangle$. By the ODE (17), $\langle \varphi(z)\varphi(0)\rangle = y^{-1/5}\Psi(x)$ lies in the kernel of the operator

$$L = y\left(p\frac{d^3}{dx^3} + \frac{3}{2}p'\frac{d^2}{dx^2} + \frac{12}{25}p''\frac{d}{dx} + \frac{12}{125} \right) ,$$

since $\frac{d}{dz} = \wp' \frac{d}{d\wp}$. Moreover, $y^{-1/5}$ lies in the kernel of the three operators

$$\frac{d}{dx} + \frac{1}{10}\frac{p'}{p}, \quad \frac{d^2}{dx^2} - \frac{1}{10}\left(\frac{11}{10}\left[\frac{p'}{p}\right]^2 - \frac{p''}{p}\right), \quad \frac{d^3}{dx^3} + \frac{1}{10}\left(\frac{231}{100}\left[\frac{p'}{p}\right]^3 - \frac{33}{10}\frac{p'p''}{p^2} + \frac{p'''}{p}\right),$$

respectively. So $L(y^{-1/5}\Psi(x)) = y^{-1/5}(L - \frac{3}{10}L_1)\Psi(x)$, where

$$L_1 = y\left(p'\frac{d^2}{dx^2} + \left(p'' - \frac{1}{10}\frac{[p']^2}{p}\right)\frac{d}{dx} + \frac{1}{3}\frac{p'''}{p} + \frac{11}{50}\frac{[p']^3}{p^2} - \frac{11}{25}\frac{p'p''}{p}\right).$$

□

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